JOURNAL OF APPROXIMATION THEORY 74, 110–121 (1993)

The Snake Theorem for Unisolvent Families

LEE L. KEENER

Department of Mathematics and Computer Science, Indiana University at South Bend, South Bend, Indiana 46634, U.S.A.

Communicated by Allan Pinkus

Received December 28, 1991; accepted in revised form

The oscillation theorem of Karlin (the snake theorem) has been extended in a number of ways by Pinkus and others. We show that the theorem is valid for (almost) arbitrary unisolvent families where there are no continuity requirements on the upper and lower bounding functions. Under an essential additional hypothesis, the theorem is also true for varisolvent families. A number of applications to uniform approximation from unisolvent families and Chebyshev spaces are considered. © 1993 Academic Press, Inc.

1. PRELIMINARIES

Karlin's oscillation theorem [4] (termed the "snake theorem" by Krein and Nudel'man [7]) has been extended in a number of ways [3, 5, 11, 14]. Pinkus has asked if the snake theorem remains valid when Chebyshev spaces are replaced by unisolvent families, and indeed has answered the question in the affirmative when the unisolvent family is extended unisolvent [11]. In this paper we will demonstrate that the theorem is valid for (almost) arbitrary unisolvent families and also, under some restrictions, for varisolvent families. Some applications to the theory of uniform approximation will also be considered. $\|\cdot\|$ will always represent the uniform norm on the compact interval $[a, b]$. The definitions of nodal zero and nonnodal zero are taken from [4].

Recall (taking the definition of Motzkin [10]) that a family Π of functions defined on $[a, b]$ is unisolvent of degree $n \geqslant 1$ if for any n values $x_1, x_2, ..., x_n$ with $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$ and arbitrary real numbers $y_1, y_2, ..., y_n$, there is an element p of Π such that $p(x_i) = y_i$, $i = 1, 2, ..., n$, and furthermore, for distinct p_1, p_2 in Π, $p_1 - p_2$ has fewer than n zeros in (a, b), where a zero x in (a, b) is counted twice if $p_1 - p_2$ does not change sign in a neighborhood of x. (Actually, this last requirement can be dropped without changing the definition.) Still further we require that for

0021-9045/93 $5.00

p in Π, $p(x) = p(x; x_1, x_2, ..., x_n; y_1, y_2, ..., y_n)$ be a continuous function of $x, y_1, y_2, ..., y_n$.

The proof that we present is an amalgam of proofs in [5] and [6], where corresponding results are developed for Chebyshev spaces. Essentially, all that is required is to show that these arguments can be modified to apply to the unisolvent setting. However, since unisolvent families need not be linear, and since [5] and [6] rely on linearity and on Chebyshev space properties at many points, the revised arguments need to be restated in detail. ([6] is not easily available in any case.) We will need the following lemma.

LEMMA 1. *Let Π be a unisolvent family of degree n, let $I_1, I_2, ..., I_m$, $1 \leqslant m \leqslant n$ be a collection of pairwise disjoint, closed subintervals of $[a, b]$ (where we further insist that $I_1 \neq [a, b]$ if $m = 1$), let g be an arbitrary element of Π, and let $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ be a given vector with real components, $|\varepsilon_i| = 1$ for $1 \leqslant i \leqslant m$ and $\varepsilon_i = -\varepsilon_{i+1}$, $1 \leqslant i \leqslant m-1$. Let $\varepsilon > 0$ be given. Then there is an element p of Π such that for each i, $1 \leqslant i \leqslant m$, $\operatorname{sgn}(p(x) - g(x)) = \varepsilon_i$ if x is in I_i, and $\|p - g\| < \varepsilon$.*

Proof. Without loss of generality, we may assume that the intervals are arranged in increasing order (using the obvious definition). Let $I_i = [a_i, b_i]$, $1 \leqslant i \leqslant m$, and assume for the moment that $m > 1$. Suppose first that $n - m$ is even. Define $c_{i+1} = \frac{1}{2}(b_i + a_{i+1})$, $1 \leqslant i \leqslant m-1$, and set $c_1 = a_1$. Choose $c_{m+1}, c_{m+2}, ..., c_n$ so that $c_m < c_{m+1} < \cdots < c_n < a_m$. For $\delta > 0$ let p be the unique element of Π such that $p(c_1) = \delta\varepsilon_1 + g(c_1)$ and $p(c_i) = g(c_i)$, $2 \leqslant i \leqslant n$. $p - g$ has zeros only at $c_2, ..., c_n$ and must change signs at these points since otherwise $p - g$ would be identically zero. Note that $\operatorname{sgn}(p(b_{m-1}) - g(b_{m-1})) = -\operatorname{sgn}(p(a_m) - g(a_m))$ since $p - g$ changes sign an odd number of times between b_{m-1} and a_m. Clearly $p - g$ has the required sign structure. Now suppose that $n - m$ is odd. Choose the c_i's as above except take $c_n = b_m$, and for $\delta > 0$, define p to be the unique element of Π for which $p(c_n) = \delta\varepsilon_m + g(c_n)$, $p(c_i) = g(c_i)$, $2 \leqslant i \leqslant n-1$, and $p(c_1) = \delta\varepsilon_1 + g(c_1)$. We claim that $p - g$ has zeros only at $c_2, ..., c_{n-1}$ on the interval $[a_1, b_m]$, and that $p - g$ changes signs at these zeros. Note first that, in any event, $p - g$ cannot have more than one zero distinct from $c_2, ..., c_{n-1}$, and such a zero must be one where $p - g$ changes sign. (Otherwise $p - g$ is, again, identically zero.) If there were such a zero in (a_1, b_m) this would force $\varepsilon_1 = \operatorname{sgn}(p(a_1) - g(a_1)) = (-1)^{n-1} \operatorname{sgn}(p(b_m) - g(b_m)) = (-1)^{n-1} \varepsilon_m$, whereas $\varepsilon_1 = (-1)^{m-1} \varepsilon_m = (-1)^n \varepsilon_m$ since $n - m$ is supposed odd, giving a contradiction. If $p - g$ failed to change signs at any one of $c_2, c_3, ..., c_{n-1}$, this would lead to a similar contradiction. It remains only to invoke the continuity property and choose $\delta > 0$ sufficiently small to guarantee that $\|p - g\| < \varepsilon$. Straightforward modifications of the above argument allow us to deal with the case $m = 1$.

2. The Snake Theorem

The following theorem is an analog of Theorem 1 in [6]. Note that we will call a zero of a function where the function changes sign a nodal zero, and a zero where there is no sign change (in some neighborhood) will be called a nonnodal zero. We say that functions u and v, defined on $[a, b]$, touch at x_0 in $[a, b]$ if there are sequences $\langle x_i \rangle$ and $\langle y_i \rangle$ in $[a, b]$ such that $x_i \to x_0$, $y_i \to x_0$, and $u(x_i) - v(y_i) \to 0$ [6].

THEOREM 2. *Let f and g be two real-valued functions defined on $[a, b]$ and let Π be a unisolvent family of functions defined on $[a, b]$, of degree n. Assume there is a function w in Π and $\varepsilon > 0$ such that $f(x) + \varepsilon \leqslant w(x) \leqslant g(x) - \varepsilon$ for all x in $[a, b]$. Then there is an element p^* in Π and points $x_1 < x_2 < \cdots < x_n$ in $[a, b]$ such that*

(a) *$f(x) \leqslant p^*(x) \leqslant g(x)$ for all x in $[a, b]$;*

(b) *f touches p^* at x_i for i odd, $1 \leqslant i \leqslant n$;*

(c) *g touches p^* at x_i for i even, $2 \leqslant i \leqslant n$.*

Furthermore, there is a function q^* in Π that satisfies conditions (a′), (b′), (c′) *obtained from* (a), (b), (c) *by replacing p^* by q^* and interchanging f and g in* (b) *and* (c). *The functions p^* and q^* are the only functions satisfying* (a), (b), (c) *and* (a′), (b′), (c′), *respectively.*

Proof. The proof closely parallels that of [6] in its general structure. We show only the existence of p^*, the q^* case being similar. The uniqueness of p^* and q^* can be shown by standard zero counting arguments. (See [4, p. 70]; these Chebyshev space arguments work equally well for unisolvent families and will be used in part of the proof of Theorem 3 below.) Let $M \subset \Pi$ be the set of all functions p in Π for which there exist n points in $[a, b]$, $z_1 < z_2 < \cdots < z_n$ and n sequences $\langle z_j^i \rangle$, $1 \leqslant j \leqslant n$ such that

$$(1) \qquad z_j^i \to z_j, \qquad 1 \leqslant j \leqslant n;$$

$$(2) \qquad f(z_j^i) \geqslant p(z_j) - \frac{1}{i}, \qquad j \text{ odd}, \qquad 1 \leqslant j \leqslant n, \qquad 1 \leqslant i < \infty;$$

$$(3) \qquad g(z_j^i) \leqslant p(z_j) + \frac{1}{i}, \qquad j \text{ even}, \qquad 2 \leqslant j \leqslant n, \qquad 1 \leqslant i < \infty.$$

The interpolation property of Π guarantees that M is nonempty. Define the functional F for each p in M by

$$F(p) = \max\{\sup\{p(x) - g(x) : x \in [a, b]\}, \sup\{f(x) - p(x) : x \in [a, b]\}\}.$$

The theorem will be established if we can show that $F(p^*) = 0$ for some $p^* \in M$. Let $\rho = \inf\{F(p) : p \in M\}$ and choose $\langle p_k \rangle$ in M so that $F(p_k) \to \rho$. We may assume (possibly by taking subsequences) that $p_k \to \bar{p}$ for some $\bar{p}$ in Π. Indeed, let $u_1, u_2, ..., u_n$ be any n distinct points in $[a, b]$ and assume $p_k(u_i) = v_i^k$, $1 \leqslant i \leqslant n$. If each of the n sequences $\langle v_i^k \rangle$ has a convergent subsequence, convergent say to $\bar{v}_i$, then there is a subsequence of $\langle p_k \rangle$ that converges to v_i at u_i, $1 \leqslant i \leqslant n$, and by the continuity property of unisolvent families, this subsequence converges to p, where p is the unique element of Π that takes the values $\bar{v}_i$ at u_i, $1 \leqslant i \leqslant n$. If $\langle v_i^k \rangle$ fails to have a convergent subsequence for some i, i.e., the sequence is unbounded, then $\langle p_k \rangle$ could not have been a minimizing sequence as was hypothesized, since then $\sup\{F(p_k) : k \geqslant 1\} = \infty$. We next show that $p \in M$. For each k, let $\langle z_{jk}^i \rangle_{i=1}^{\infty}$ and z_{jk}, for $j = 1, 2, ..., n$ be the n sequences and sequence limits associated with p_k from the definition of M. Assume without loss of generality that $|z_{jk}^i - z_{jk}| \leqslant 1/i$ for all k and all j. Again by taking subsequences if necessary, we may assume that $z_{jk} \to \bar{z}_j$ for $1 \leqslant j \leqslant n$, $\bar{z}_1 \leqslant \bar{z}_2 \leqslant \cdots \leqslant \bar{z}_n$ are in $[a, b]$. Finally, we may assume without loss of generality that $|z_{jk} - \bar{z}_j| \leqslant 1/k$ for all k and j. For each j we find a sequence $\langle y_j^m \rangle_{m=1}^{\infty}$ and a sequence $\langle s_j^m \rangle_{m=1}^{\infty}$ such that $s_j^m \to 0$ and $y_j^m \to \bar{z}_j$ and either $f(y_j^m) \geqslant \bar{p}(\bar{z}_j) - s_j^m$ if j is odd or $g(y_j^m) \leqslant \bar{p}(\bar{z}_j) + s_j^m$ if j is even. Indeed, define $\langle y_j^m \rangle_{m=1}^{\infty}$ by $y_j^m = z_{jm}^m$ for all m. Then

$$|z_{jm}^m - \bar{z}_j| \leqslant |z_{jm}^m - z_{jm}| + |z_{jm} - \bar{z}_j| \leqslant \frac{1}{m} + \frac{1}{m} = \frac{2}{m}$$

so $y_j^m \to \bar{z}_j$ for each j. Suppose j is odd. Then

$$f(z_{jm}^m) \geqslant p_m(z_{jm}) - \frac{1}{m} = \bar{p}(\bar{z}_j) - [(\bar{p}(\bar{z}_j) - \bar{p}(z_{jm})) + (\bar{p}(z_{jm}) - p_m(z_{jm}))] - \frac{1}{m}.$$

Setting $s_j^m = [(\bar{p}(\bar{z}_j) - \bar{p}(z_{jm})) + \bar{p}(z_{jm}) - p_m(z_{jm}))] + 1/m$, we observe that $s_j^m \to 0$ for each j by the (uniform) convergence of $\langle p_m \rangle$ to $\bar{p}$ and the continuity of $\bar{p}$. A similar treatment is used for j even. By taking subsequences if necessary we may find sequences that satisfy the conditions in the definition of M. It is clear from the continuity of p and the hypothesis on the existence of the function w that in fact we have $\bar{z}_1 < \bar{z}_2 < \cdots < \bar{z}_n$. Thus $\bar{p} \in M$. If $\rho = F(\bar{p}) = 0$ we are done, so suppose to the contrary that $F(\bar{p}) > 0$. Using standard arguments and recalling again the existence of w we can find n disjoint intervals $I_j = [a_j, b_j]$, $j = 1, 2, ..., n$, and a $\delta > 0$ such that $\bar{z}_j \in I_j$ and $\bar{p}(x) \leqslant g(x) - \delta$ if $x \notin \bigcup \{I_j\colon j \text{ is even}\}$ and $\bar{p}(x) \geqslant f(x) + \delta$ if $x \notin \bigcup \{I_j\colon j \text{ is odd}\}$. Define the n-tuple $(\sigma_1, \sigma_2, ..., \sigma_n)$ by

$$\sigma_j = \begin{cases} 1 & \text{if } j \text{ is even and } \bar{p}(x) > g(x) \text{ for some } x \in I_j; \\ -1 & \text{if } j \text{ is odd and } \bar{p}(x) < f(x) \text{ for some } x \in I_j; \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_j = \frac{1}{2}(b_j + a_{j+1})$ for $j = 1, 2, ..., n-1$. Let $r_1, r_2, ..., r_{m-1}$ be those elements among the d_j's, arranged in increasing order, for which $|\sigma_j - \sigma_{j+1}| \neq 1$ and let K_i, $2 \leqslant i \leqslant m-1$ be chosen so that K_i is the smallest interval that contains all of the I_j intervals lying between r_{i-1} and r_i, with appropriately modified definitions for K_1 and K_m. (We assume $m > 1$ here. A straightforward modification handles the $m = 1$ case.) By Lemma 1, for a given $\varepsilon > 0$, there is an element p_ε in Π such that $\bar{p} - p_\varepsilon$ alternates in sign on $K_1, K_2, ..., K_m$, $\|\bar{p} - p_\varepsilon\| < \varepsilon$ and has a stipulated signum value on any single given K_i. In fact, by supposition, for some $j = j^*$, $\sigma_{j^*} \neq 0$. So let us assume that p_ε is chosen so that $\operatorname{sgn}(\bar{p}(\bar{z}_{j^*}) - p_\varepsilon(\bar{z}_{j^*})) = \sigma_{j^*}$. For sufficiently small $\varepsilon > 0$, p_ε is such that $0 < \sup\{p_\varepsilon(x) - g(x)\colon x \in I_{j^*}\} < \sup\{\bar{p}(x) - g(x)\colon x \in I_{j^*}\}$ if j^* is even or $0 < \sup\{f(x) - p_\varepsilon(x)\colon x \in I_{j^*}\} < \sup\{f(x) - \bar{p}(x)\colon x \in I_{j^*}\}$ if j^* is odd. From the definitions of p_ε and $(\sigma_1, \sigma_2, ..., \sigma_n)$ we may deduce, by working step by step from I_{j^*} through the intervals to the left and right of I_{j^*}, that for sufficiently small $\varepsilon > 0$, inequalities of the above form hold for all I_j with $\sigma_j \neq 0$, that $p_\varepsilon \in M$, that $p_\varepsilon(x) < g(x)$ if $x \notin \bigcup \{I_j\colon j \text{ is even}\}$ and $p_\varepsilon(x) > f(x)$ if $x \notin \bigcup \{I_j\colon j \text{ is odd}\}$. These facts together imply that $F(p_\varepsilon) < \rho$, a contradiction, and the theorem follows.

The next theorem is the analog of Theorem 2 of [6].

THEOREM 3. *Let f, g be two real-valued functions defined on $[a, b]$ and let Π be a unisolvent family of functions of degree n, defined on $[a, b]$. We make the further technical assumption (Assumption* A) *that every element of Π is bounded away from f on some subinterval of $[a, b]$ and from g on some subinterval of $[a, b]$. Assume that Q, defined by $Q = \{q \in \Pi\colon f(x) \leqslant q(x) \leqslant g(x)$ for all $x \in [a, b]\}$ is nonempty and that f and g do not touch on $[a, b]$. Then the following are equivalent:*

(2.1) *Q is a singleton.*

(2.2) *There exists a q^* in Q that oscillates $n+1$ times between f and g (i.e., q^* alternately touches f and g at $n+1$ points and lies between f and g).*

(2.3) *There is no q in Q for which $\varepsilon > 0$ can be found such that $f + \varepsilon \leqslant q \leqslant g - \varepsilon$ on $[a, b]$.*

Proof. (2.2) → (2.1). Suppose q_1 is an element of Q distinct from q^* (whose existence is asserted by (2.2)). Then $q_1 - q^*$ has at least n zeros counting nonnodal zeros twice. Indeed, assume that $a \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_{n+1} \leqslant b$ are the $n+1$ points in $[a, b]$ at which q^* alternates, and further assume without loss of generality that q^* touches the upper function g at the points with even subscripts. Then $q_1(\alpha_i) \geqslant q^*(\alpha_i)$ for i odd and $q_1(\alpha_i) \leqslant q^*(\alpha_i)$ for i even, since $q_1(\alpha_i) < q^*(\alpha_i)$ for any odd i or $q_1(\alpha_i) > q^*(\alpha_i)$ for any even i would imply that q^* did not touch f (respectively g) at α_i. Assume also that $n = 2m$, i.e., n is even. The argument that follows is a variation on that found in Karlin and Studden [4, pp. 70–71]. For the purposes of the proof, we say that a function h has a special zero at t_0 in a closed interval $[c, d]$ if $h(t_0) = 0$ and either $t_0 \in (c, d)$ (and for counting purposes is counted twice if nonnodal) or $t_0 = c$, $h(t) \geqslant 0$ for $t \in (c, c+\delta)$ or $t_0 = d$, $h(t) \geqslant 0$ for $t \in (d-\delta, d)$. Let z_j be the number of special zeros of $q^* - q_1$ in $[\alpha_{2j+1}, \alpha_{2j+3}]$, $j = 0, 1, ..., m-1$. Then $z_j \geqslant 2$ for $j = 0, 1, ..., m-1$ since $q^*(\alpha_{2j+1}) \leqslant q_1(\alpha_{2j+1})$, $q^*(\alpha_{2j+2}) \geqslant q_1(\alpha_{2j+2})$, $q^*(\alpha_{2j+3}) \leqslant q_1(\alpha_{2j+3})$, so there are either two distinct special zeros in $[\alpha_{2j+1}, \alpha_{2j+3}]$ or else there is a nonnodal zero (counted twice) at α_{2j+2}. Thus $\sum_{l=0}^{m-1} z_j \geqslant 2m = n$. But $\sum_{j=0}^{m-1} z_j$ is no greater than the number of zeros of $q^* - q_1$ on $[a, b]$ counting nonnodal zeros twice, nodal zeros once. (We note that nonnodal zeros that occur at α_{2j+1} for $j = 1, 2, ..., m-1$ are counted twice in $\sum z_j$.) A straightforward modification of the above argument handles the case $n = 2m+1$. Thus $q^* - q_1$ must be identically zero since Π is unisolvent, contradicting the supposition of distinctness.

(2.3) → (2.2). Let $\bar{q}$ be in Q and suppose (2.2) is false, so that $\bar{q}$ does not oscillate $n+1$ times. Let $a \leqslant \beta_1 < \beta_2 < \cdots < \beta_m \leqslant b$ be a set of m points on which $\bar{q}$ oscillates a maximal number of times, so $m \leqslant n$. Assume without loss of generality that $\bar{q}$ touches g at points β_i with even subscripts. Define, for each i, real numbers ℓ_i and u_i by

$$\ell_i = \inf\{x : a \leqslant x \leqslant \beta_i,\ \bar{q} \text{ touches } g \text{ at } x, \text{ and there is no } y \in [x, \beta_i] \text{ such that } \bar{q} \text{ touches } f \text{ at } y\}$$

$$u_i = \sup\{x : \beta_i \leqslant x \leqslant b,\ \bar{q} \text{ touches } g \text{ at } x, \text{ and there is no } y \in [\beta_i, x] \text{ such that } \bar{q} \text{ touches } f \text{ at } y\}$$

provided i is even. If i is odd, define u_i, ℓ_i similarly, reversing the roles of f and g. Since f does not touch g, $u_i < \ell_{i+1}$ for $i = 1, 2, ..., m-1$. Furthermore, if t is any point at which $\bar{q}$ touches either f or g, then t is in $\bigcup_{i=1}^{m} [\ell_i, u_i]$ (as a result of the maximality of the β_i's). Let p_δ be an element of Π that is strictly greater than $\bar{q}$ on $[\ell_1, u_1]$ and alternately strictly less than and strictly greater than $\bar{q}$ on $[\ell_2, u_2]$, ..., $[\ell_m, u_m]$, and such that $\|\bar{q} - p_\delta\| < \delta$. Such a function exists by Lemma 1. Note that if $m = 1$, $[\ell_1, u_1] \neq [a, b]$ as a consequence of Assumption A made in the hypotheses. The necessity of the assumption here is related to the "constant error curve problem" of varisolvent uniform approximation. Assumption A is a convenient hypothesis to make that avoids this possible pathology; it can be substantially weakened or perhaps removed for the unisolvent case. See [9] for a complete discussion of the problem. Using standard arguments, it follows that for sufficiently small $\delta > 0$, p_δ touches neither f nor g at any point in $[a, b]$. Indeed, for sufficiently small $\delta > 0$, an $\varepsilon > 0$ can be found such that $f + \varepsilon \leqslant p_\delta \leqslant g - \varepsilon$ on $[a, b]$, since if p_δ does not touch f or g, it must be bounded away from each. That is, (2.2) does not hold and the result follows by contraposition.

(2.1) $\rightarrow$ (2.3). This implication follows directly from the observation that given an element of a unisolvent family, there is another element arbitrarily close to it (in the uniform norm).

Theorem 3 is implicit in Pinkus's work though with stronger hypotheses. The argument given here is fundamentally different from his, the latter being a limit argument.

We may combine Theorems 2 and 3 to obtain

THEOREM 4. *Let f, g be two real-valued functions defined on $[a, b]$ and assume that f, g do not touch on $[a, b]$. Let Π be a unisolvent family of degree n of continuous functions on $[a, b]$, where Π satisfies Assumption* A *of Theorem* 3. *Assume $Q \equiv \{q \in \Pi : f \leqslant q \leqslant g$ on $[a, b]\} \neq \varnothing$. Then there is an element p^* in Π and points $x_1 < x_2 < \cdots < x_n$ in $[a, b]$ such that*

(a) *$f(x) \leqslant p^*(x) \leqslant g(x)$ for all x in $[a, b]$;*

(b) *f touches p^* at x_i for i odd, $1 \leqslant i \leqslant n$;*

(c) *g touches p^* at x_i for i even, $2 \leqslant i \leqslant n$.*

Furthermore, there is a function g^* in Π that satisfies conditions (a′), (b′), (c′) *obtained from* (a), (b), (c) *by replacing p^* by q^* and interchanging f and g in* (b) *and* (c). *The functions p^* and q^* are the only functions in Π satisfying* (a), (b), (c) *and* (a′), (b′), (c′), *respectively.* If $Q_\varepsilon \equiv \{q \in \Pi : f + \varepsilon \leqslant q \leqslant g - \varepsilon$ *on* $[a,b]\} = \varnothing$ *for all* $\varepsilon > 0$, *then the functions p^* and q^* are equal.*

Proof. The result is true by Theorem 3 if Q is a singleton. If not, then $Q_\varepsilon \neq \varnothing$ for some $\varepsilon > 0$, again by Theorem 3, and the hypotheses of Theorem 4 agree with those of Theorem 2.

3. Applications to Uniform Approximation

Definition. Let $S_M \subset C[a, b]$, and let $M: S_M \to [0, \infty]$ be a given extended real-valued functional and define $\rho(M, \sigma) \equiv M^{-1}([0, \sigma])$. M is said to be norm-like (an NL-functional) if $\bar{\sigma} = \inf\{M(p): p \in S_M\} > 0$ and there exist two families of real-valued functions defined on $[a, b]$, $F_M = \{f_\sigma: \sigma \geqslant 0\}$ and $G_M = \{g_\sigma: \sigma \geqslant 0\}$ with the following properties:

(3.1) f_σ and g_σ do not touch for $\sigma \geqslant \bar{\sigma}$;

(3.2) for σ finite, $p \in \rho(M, \sigma)$ iff $f_\sigma \leqslant p \leqslant g_\sigma$ and $p \in S_M$;

(3.3) for σ finite, $M(p) = \sigma$ only if there is no $\varepsilon > 0$ such that $f_\sigma + \varepsilon \leqslant p \leqslant g_\sigma - \varepsilon$.

Theorem 5. *Let Π be a unisolvent family of continuous functions on $[a, b]$ of degree n, and let M be an NL-functional with $S_M = \Pi$. If p^* minimizes M over Π and F_M and G_M are given, then p^* oscillates $n+1$ times between f_{σ^*} and g_{σ^*}, for some $\sigma^* \geqslant 0$. Conversely, if for some choice of F_M and G_M, p^* in Π oscillates $n+1$ times between f_{σ^*} and g_{σ^*}, for some $\sigma^* \geqslant 0$, then p^* minimizes M over Π.*

Proof. Assume p^* oscillates $n+1$ times between f_{σ^*} and g_{σ^*}. Then $Q^* = \{p \in \Pi: f_{\sigma^*} \leqslant p \leqslant g_{\sigma^*}\}$ is a singleton by Theorem 3 (where by (3.2), $\sigma^* \geqslant \bar{\sigma}$ so by (3.1) f_{σ^*} and g_{σ^*} do not touch). Therefore, for all $p \in \Pi - \{p^*\}$, $p \notin \rho(M, \sigma^*)$ by (3.2), i.e., $M(p) > \sigma^*$, whereas $M(p^*) \leqslant \sigma^*$ by (3.2), so p^* uniquely minimizes M. Conversely, suppose p^* minimizes M. Let $M(p^*) = \bar{\sigma} = \inf\{M(p): p \in \Pi\}$. Then there is no pair $\varepsilon > 0$ and $p \in \Pi$ such that $f_{\bar{\sigma}} + \varepsilon \leqslant p \leqslant g_{\bar{\sigma}} - \varepsilon$, since such a p would imply by (3.3) that $M(p) \neq \bar{\sigma}$, hence $M(p) < \bar{\sigma}$, contradicting the definition of p^*. By Theorem 2, this implies p^* oscillates $n+1$ times. By (3.2), it is clear that $\sigma^* \geqslant M(p^*)$.

In the following applications, the verification that M is an NL-functional is routine. Each was considered by Pinkus [11]. We reproduce them, in the setting of the NL-functional. The continuity properties that must be assumed on the approximated or bounding functions are in some cases, weaker than have heretofore been given in the literature. Π is, in each instance, a unisolvent family of degree n. We assume Assumption A holds here, though in light of the known (slightly weaker) results, this is almost surely an expendable hypothesis.

Application 1. Consider uniform approximation of a given q on $[a, b]$ from Π. Define $M(p) = \|p - q\|$. Then take $f_\sigma = q - \sigma$, $g_\sigma = q + \sigma$. If q is continuous, f_σ and g_σ cannot touch for $\sigma > 0$. The classic alternation theorem [1] is the immediate result where we assume $q \notin \Pi$. But notice that f_σ and g_σ possibly may not touch for $\sigma \geqslant \bar{\sigma}$ even if q is discontinuous.

Application 2. Consider uniform approximation of a continuous q on $[a, b]$ from Π with restricted ranges, where we approximate from $\Pi' = \{p \in \Pi : \ell(x) \leqslant p(x) \leqslant u(x)$ for all $x \in [a, b]\}$. We assume ℓ and u do not touch. Define $M(p) = \|p - q\|$ if $\ell \leqslant p \leqslant u$ and $M(p) = \infty$ otherwise. Take $f_\sigma = \max\{\ell, q - \sigma\}$, $g_\sigma = \min\{u, q + \sigma\}$. It is easy to see that for $\sigma > 0$, if ℓ and u do not touch on $[a, b]$ then f_σ and g_σ also do not touch. The alternation theorem that results is well known [13], but note that the hypotheses on ℓ and u are weaker than those in [13].

Application 3. Consider simultaneous uniform approximation of q_1, q_2 on $[a, b]$ from Π using the functional $M(p) = \max\{\|p - q_1\|, \|p - q_2\|\}$. Define $q_u = \max\{q_1, q_2\}$ and $q_\ell = \min\{q_1, q_2\}$. Define $f_\sigma = q_u - \sigma$, $g_\sigma = q_\ell + \sigma$. It is possible for f_σ and g_σ to touch if Π contains functions with straddle points (see [2]), but if q_1 and q_2 are continuous and there are no straddle points, f_σ and g_σ cannot touch for $\sigma \geqslant \bar{\sigma}$ and Dunham's result [2] follows immediately.

Other applications include approximation with constraints outside the interval of approximation (see [8], for example) and approximation by reciprocals of elements of a Chebyshev space. Additional applications, such as approximation by trigonometric polynomials on intervals longer than 2π, "maximin" approximation and "modular" approximation will be examined in future papers.

4. Extension to Varisolvent Families

Varisolvent families were first defined and studied by Rice. (See [12] for a complete discussion.) We take the following definition:

Definition. Let Π be a family of real-valued functions, continuous on $[a, b]$. If for each $p \in \Pi$ there is a number $m(p)$ (the degree of p) such that given $x_1, x_2, ..., x_{m(p)}$ with $a \leqslant x_1 < \cdots < x_{m(p)} \leqslant b$, and $\varepsilon > 0$ it is possible to find $\delta > 0$ such that

(a) if $q \in \Pi$, $q \neq p$, then $q - p$ has at most $m(p) - 1$ zeros;

(b) $|y_j - p(x_j)| < \delta$ for $j = 1, 2, ..., m(p)$ implies the existence of $p_1 \in \Pi$ such that $p_1(x_j) = y_j$, $j = 1, 2, ..., m(p)$ and $\|p - p_1\| < \varepsilon$.

If, in addition, $1 \leqslant m(p) \leqslant D$ for some integer D and all $p \in \Pi$, then Π is said to be varisolvent.

Under appropriate conditions, the snake theorem is valid for varisolvent families.

THEOREM 6. *Let f and g be two bounded, real-valued functions defined on $[a, b]$ and let Π be a varisolvent family of functions defined on $[a, b]$. Assume there is a function w_1 in Π and $\varepsilon > 0$ such that $f(x) + \varepsilon \leqslant w_1(x) \leqslant g(x) - \varepsilon$ for all x in $[a, b]$. Assume also that there is a function w_2 in Π and points $\alpha_1 < \alpha_2 < \cdots < \alpha_{m(w_2)}$ in $[a, b]$ such that $w_2(\alpha_i) \leqslant f(\alpha_i)$ for i odd and $w_2(\alpha_i) \geqslant g(\alpha_i)$ for i even, $1 \leqslant i \leqslant m(w_2)$. Assume that Assumption* A *holds. Finally, assume that Π is boundedly sequentially compact. (I.e., the intersection of Π with any closed ball is sequentially compact.) Then there is an element p^* in Π and points $x_1 < x_2 < \cdots < x_{m(p^*)}$ in $[a, b]$ such that*

(a) *$f(x) \leqslant p^*(x) \leqslant g(x)$ for all x in $[a, b]$;*

(b) *f touches p^* at x_i for i odd, $1 \leqslant i \leqslant m(p^*)$;*

(c) *g touches p^* at x_i for i even, $2 \leqslant i \leqslant m(p^*)$.*

Furthermore, if "odd" and "even" are interchanged in the hypothesis on w_2 then there is a function q^* in Π that satisfies conditions (a′), (b′), (c′) *obtained from* (a), (b), (c) *by replacing p^* by q^* and interchanging f and g in* (b) *and* (c). *The functions p^* and q^* are the only functions satisfying* (a), (b), (c) *and* (a′), (b′), (c′), *respectively.*

Proof. The statement of this theorem is similar to that of Theorem 2, but with added hypotheses. The proof follows closely the proof of that theorem. There are two fundamental problems: the inability to guarantee the existence of an interpolating element of Π for any given set of points and the varying degree. The former problem appears at the point in the proof where M is claimed to be nonempty. The hypothesis concerning w_2 explicitly guarantees that $w_2 \in M$. The problem appears again when $\langle p_k \rangle$ is claimed to converge to $\bar{p} \in \Pi$. But since f and g are bounded, $\langle p_k \rangle$ must be bounded and hence $\bar{p} \in \Pi$ by sequential compactness. The latter problem can be dealt with by noting that we may assume, by passing to subsequences if necessary that $m(p_k) = m^*$ (where m^* is the largest degree appearing in the original sequence an infinite number of times). By Theorem 7-1 in [12], $m(\bar{p}) \leqslant m^*$. For each k, $p_k - w_1$ has $m(p_k) - 1 = m^* - 1$ simple zeros by the intermediate value theorem. Also, there is a p_{k^*} for which $\|\bar{p} - p_{k^*}\| < \varepsilon$, implying $\bar{p} - w_1$ also has $m^* - 1$ simple zeros. By varisolvency, $m(\bar{p}) \geqslant m^*$ and in fact we have $m(\bar{p}) = m^*$. The remainder of the proof mimics that of Theorem 2 with m^* replacing n (in Lemma 1 as well).

COROLLARY 7. *The set $\Pi = \{\sum_{i=1}^{n} a_i \phi_i(x) / \sum_{i=1}^{m} b_i \Psi_i(x) : \sum_{i=1}^{m} b_i \Psi_i(x) > 0$ for $x \in [a, b]$, $\sum_{i=0}^{m} (b_i^2) = 1\}$ (generalized rationals) where the Ψ_i's and*

ϕ_i's are analytic, is boundedly sequentially compact. So, if in addition, Π is hypothesized to be varisolvent, then Π satisfies the conditions of Theorem 6. *In particular, ordinary rationals satisfy those conditions.*

Proof. By Lemma 9–1 in [12], if a subset of Π is bounded then the coefficients of the elements of the subset, $\{a_i\}$ and $\{b_i\}$, are uniformly bounded. Thus if for some $\mathcal{M}$, $\langle p_k \rangle$ is a sequence from Π with $\|p_k\| \leqslant \mathcal{M}$ for all k, there is a subsequence of $\langle p_k \rangle$ such that the coefficients of the p_k functions converge to a coefficient vector $\bar{c} = (\bar{a}_1, \bar{a}_2, ..., \bar{a}_n, \bar{b}_1, \bar{b}_2, ..., \bar{b}_m)$. Using the standard argument for the existence of best rational approximations (for example, [1, p. 155]), the function $\bar{p}$ corresponding to the vector $\bar{c}$ is in Π and $\|\bar{p}\| \leqslant \mathcal{M}$; i.e., Π is boundedly sequentially compact. Ordinary rationals are known to be varisolvent [12].

Remark. The hypothesis in Theorem 6 about the existence of w_2 cannot be removed. Consider for example the situation where $f(x) < 0$ and $g(x) > 0$ for all x in $[a, b]$. Any element p of $\Pi = R_m^n$ (ordinary rationals with numerator of degree $\leqslant n$ and denominator of degree $\leqslant m$) can oscillate at most $n^* + 1$ times between f and g on $[a, b]$ (where n^* is the degree of the numerator of p) since a greater number of oscillations would force p to have at least $n^* + 1$ zeros, making the numerator of p and hence p itself the zero function. But it is well known that if $p \not\equiv 0$, $m(p) = n + m - d + 1$, where $d = \min\{r, s : a_{n-r} \neq 0, b_{m-s} \neq 0\}$. So we have $d \leqslant n - n^*$ and $m(p) = n + m - d + 1 \geqslant n + m - (n - n^*) + 1 = m + n^* + 1 > n^* + 1$ provided we choose $m \geqslant 1$.

5. Conclusion

Theorem 4 generalizes somewhat the version of the snake theorem in [5]. In addition, Theorems 4 and 5 together clearly show the relation between the snake theorem and the alternation theorems of uniform approximation. The rather general Theorem 5 can be applied to a variety of approximation settings. In some cases, these applications provide new results or extensions to known results, often by virtue of the lack of continuity requirements on the f and g functions. We comment that Theorem 3 remains valid if we replace $[a, b]$ by a compact subset of $[a, b]$ containing at least $n + 1$ points. It would be useful to apply Theorem 6 to problems of uniform approximation by varisolvent families. Unfortunately, the hypothesis on the existence of the function w_2 causes certain difficulties. Finally, it appears that the second Remes algorithm may be adapted to minimize NL-functionals. A future paper will deal with this adaptation.

REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
2. C. DUNHAM, Simultaneous Chebyshev approximation of functions of an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472–477.
3. B. GOPINATH AND R. P. KURSHAN, The oscillation theorem for Tchebycheff spaces of bounded functions, and a converse, *J. Approx. Theory* **21** (1977), 151–173.
4. S. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems with Applications in Analysis and Statistics," Interscience, New York, 1966.
5. L. KEENER, A new proof of the snake theorem, *Canad. Math. Bull.* **25** (2) (1982), 234–237.
6. L. KEENER, "The Snake Theorem and Uniform Approximation," Indiana University at South Bend, Department of Mathematics and Computer Science, Technical Report M061190-5, 1990.
7. M. G. KREIN AND A. A. NUDEL'MAN, "The Markov Moment Problem and Extremal Problems," Vol. 50, Translations of Mathematical Monographs, A.M.S., Providence, RI, 1977.
8. P. J. LAURENT, Approximation uniforme de fonctions continues sur un compact avec constraints de type inégalité, *Rev. Fr. Automat. Informat. Recherche Opér.* **1** (1967), 81–95.
9. W. H. LING AND J. E. TORNGA, The constant error curve problem for varisolvent families, *J. Approx. Theory* **11** (1974), 54–72.
10. T. S. MOTZKIN, Approximation by curves of a unisolvent family, *Bull. Amer. Math. Soc.* **55** (1949), 789–793.
11. A. PINKUS, Applications of representation theorems to problems of Chebyshev approximation with constraints, *in* "Studies of Spline Approximation Theory" (S. Karlin, C. Micchelli, A. Pinkus, and I. J. Schoenberg, Eds.), Academic Press, New York, 1976.
12. J. R. RICE, "The Approximation of Functions, Vol. 2: Nonlinear and Multivariate Theory," Addison-Wesley, Reading, MA, 1969.
13. G. D. TAYLOR, On approximation by polynomials having restricted ranges, I, *SIAM J. Numer. Anal.* **5** (1968), 258–268.
14. R. ZIELKE, "Discontinuous Čebyšev Systems," Lecture Notes in Mathematics, No. 707, Springer-Verlag, Berlin/New York, 1979.

Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis